The Schrijver System of the Flow Cone in Series-Parallel Graphs

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Mots-clés: Total Dual Integrality; Schrijver System; Series-Parallel Graph.

Given an undirected graph \( G = (V, E) \), a flow of \( G \) is a couple \((C, e)\) with \( C \) a circuit of \( G \) and \( e \) an edge of \( C \). In a flow \((C, e)\), the edge \( e \) represents a demand and \( C \setminus e \) represents the path satisfying this demand. For a subset \( S \) of \( E \), let \( \chi^S \) denote the incidence vector of \( S \), that is the \( 0/1 \) vector such that \( \chi^S_e = 1 \) if \( e \) is in \( S \), and 0 otherwise. With an abuse of language, we say that the incidence vector of a flow \((C, e)\) is the \( 0/\pm1 \) vector \( \chi^C - \chi^e \). The flow cone of \( G \) is the cone generated by the flows of \( G \) and the unit vectors of \( \mathbb{R}^E \), \( \chi^e \) for every \( e \) in \( E \).

The concept of total dual integrality dates back to the works of Edmonds, Giles, and Pulleyblank in the late 70's of the last century. This concept is strongly connected to min-max relations in combinatorial optimization problems. Let us be given a linear system, and its linear programming duality equation:

\[
\min \{ cx : Ax \geq b \} = \max \{ yb : yA = c, y \geq 0 \} \tag{1}
\]

\( Ax \geq b \) is totally dual integral (TDI) if, for each integer vector \( c \) for which the minimum in \( (1) \) is finite, there exists an integer optimal solution for the maximum in \( (1) \). A system \( Ax \geq b \) is said box-TDI if the system \( Ax \geq b; \ell \leq x \leq u \) is TDI for all rational \( \ell \) and \( u \). General properties of such systems can be found in [6, Chap. 22].

Every polyhedron \( P \) can be described by a TDI system \( Ax \leq b \). Moreover, if \( P \) is integer, \( A \) and \( b \) can be chosen integer. Schrijver [5] showed that, if \( P \) is full-dimensional, there exists a unique minimal integer TDI system describing \( P \). This system is known as the Schrijver System of \( P \). We are generally interested in integer TDI systems because they lead to min-max relations between combinatorial objects. In addition, Schrijver systems give the strictest min-max relations of this kind.

In this paper, we are interested in the Schrijver system of the flow cone in series-parallel graphs. A graph is series-parallel if it does not contain \( K_4 \) as a minor [4].

A cut \( \delta(W) \) is the set of edges having exactly one endpoint in a subset \( W \) of \( V \). Given a partition \( \{V_1, \ldots, V_k\} \) of \( V \), the set of edges having endpoints in two distinct \( V_i \)'s is called multicut. The cut cone of \( G \) is the cone generated by the incidence vectors of the cuts of \( G \). For this cone is already known a TDI system with integer coefficients [7, 8]. Moreover it was recently proved by Cornaz, Grappe, and Lacroix [2] that this system is box-TDI when the graph is series-parallel. When \( G \) has no \( K_5 \)-minor, the flow cone of \( G \) is the polar of the cut cone and it is described by \( x(D) \geq 0 \), for all cuts \( D \) of \( G \) [8].

Contribution The goal of this paper is to answer the question raised by Chervet, Grappe, and Robert [3]. Quoting them, they “leave open the question of finding a box-TDI system with integer coefficients, which exists by [6, Theorem 22.6(i)] and [1, Corollary 2.5].”
We first prove that

\[ x(M) \geq 0 \quad \text{for all multicuts } M \text{ of } G, \]  

(2)
is a TDI system describing the flow cone if and only if the graph is series-parallel. As the flow cone is a box-TDI polyhedron for such graphs \([3]\), this implies that System (2) is a box-TDI system if and only if the graph is series-parallel. We then refine this result by providing the corresponding Schrijver system.

Références