An extended formulation for Location-Routing Problem

Pedro Liguori	extsuperscript{1}, A. Ridha Mahjoub	extsuperscript{1}, Ruslan Sadykov	extsuperscript{2}, Eduardo Uchoa	extsuperscript{3}

	extsuperscript{1} LAMSADE, Université Paris-Dauphine PSL Research University, Paris, France
{pedro-henrique.pereira-vargas-liguori,ridha.mahjoub}@dauphine.fr

	extsuperscript{2} Inria Bordeaux Sud-Ouest, Talence, France
Ruslan.Sadykov@inria.fr

	extsuperscript{3} LOGIS, University Federal Fluminense, Niteroi, Brazil

Mots-clés : Branch-and-Cut-and-Price, column generation, location-routing problem

1 Introduction

In this work, we consider the standard \textit{Capacitated Location-Routing Problem} (LRP), which is defined on a weighted undirected graph $G = (I \cup J, E \cup F)$, where $I$ is the set of possible depot locations, $J$ is the set of customers, the edge $E = J \times J$ and $F = I \times J$ represent cheapest paths, with costs $c : E \cup F \to \mathbb{R}_+$, between pairs of vertices. Additionally, we associate capacities $W(i)$ and installation cost $f(i)$, for $i \in I$, and demands $d(j)$ for $j \in J$. In this context, a \textit{route} is an elementary cycle in $G$ containing exactly one depot in $I$ and a subset of the customers $J$. Therefore, a feasible solution to LRP is a set of routes such that: (i) each customer belongs to exactly one route; (ii) the sum of the customers’ demands in a route does not exceed $Q$, the vehicle capacity; and, (iii) the sum of the customers’ demands in all routes associated with depot $i \in I$ does not exceed $W(i)$. Then, the objective is to find a feasible solution such that the route cost, given by the sum of the costs of the edges in each route, and the installation costs of depots used in the solution, are minimized.

As observed by Contardo \textit{et al.} [1], the LRP generalizes two important NP-hard problems: the \textit{Capacitated Vehicle Routing Problem} (CVRP) and the \textit{Capacitated Facility Location Problem} (CFLP).

2 Formulation with an exponential number of variables

For every $i \in I$, let $y(i)$ be a binary variable equal to 1 iff the depot $i$ is opened. For every edge $(i, j) \in F$, let $z(ij)$ be a binary variable equal to 1 iff the customer $j$ is served by depot
$i$. Let $\Omega(i)$ be the set of all feasible routes associated with depot $i \in I$. Given $\omega \in \Omega(i)$, let $a_\omega(e)$ be a coefficient indicating how many times the edge $e \in E \cup F$ is traversed by the route $\omega$. Finally, let $\lambda(\omega)$ be a binary variable equal to 1 iff the route $\omega \in \Omega(i)$ is used in the solution. Then the LRP can be formulated as

$$
\min \left\{ \sum_{i \in I} \sum_{\omega \in \Omega(i)} \left( \sum_{e \in E \cup F} c(e) a_\omega(e) \right) \lambda(\omega) + \sum_{i \in I} f(i) y(i) \right\}
$$

subject to

$$
\sum_{i \in I} z(ij) = 1 \quad \forall j \in J,
$$

$$
\sum_{\omega \in \Omega(i)} \sum_{e \in \delta(j)} a_\omega(e) \lambda(\omega) = 2z(ij), \quad \forall i \in I, j \in J,
$$

$$
z(ij) \leq y(i) \quad \forall i \in I, j \in J,
$$

$$
\sum_{j \in J} d(j) z(ij) \leq W(i) y(i) \quad \forall i \in I,
$$

together with non-negativity and integrality constraints for all variables. Constraints (2) guarantee that every customer is served by exactly one depot. Constraints (3) assure that, if customer $j$ is served by depot $i$, then there exists a route leaving depot $i$ and passing through customer $j$. Constraints (4) imply that customers can only be serviced by an opened depot and constraints (5) guarantee depot capacities.

### 3 Algorithm and results

We have extended the Branch-and-Cut-and-Price Algorithm of Sadykov et al. [2] to solve formulation (2)–(5) reinforced by some valid inequalities. Preliminary results showed that our algorithm could solve to optimality, for the first time, 12 open instances of the most difficult classes $\mathcal{F}_2$ and $\mathcal{F}_4$. These instances, containing up to 200 customers and 10 depot locations, could not be solved by the state-of-the-art approach by Contardo et al. [1]. The only remaining open instance for class $\mathcal{F}_2$ is now 200-10-3b.

### Références
