A new approach for optimizing over the efficient set of convex multi-objective optimization problems

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Multi-objective optimization (also referred to as vector or multi-criteria optimization) deals with the situations where one wishes to minimize several conflicting objectives. Formally, multi-objective optimization can be formulated as follows

\[
\min_x (f_1(x),\ldots,f_p(x))
\]

where \( p \geq 2 \), \( x = (x_1,\ldots,x_n) \) is the vector of decision variables, \( X \subseteq \mathbb{R}^n \) represent the feasible set and \( f_k : \mathbb{R}^n \rightarrow \mathbb{R} \) for \( k = 1,\ldots,p \) are the objective functions.

Unlike the case with a single objective, the optimal solution for multi-objective optimization is in general not a single point but a set of solutions where each solution represents a compromise between the different objectives. To solve this type of problems, one has to find all efficient solutions or weakly efficient solutions in the sense of the following definitions \cite{1}. A feasible point \( \hat{x} \in X \) is an efficient (Pareto optimal, nondominated) solution for problem (1) if and only if there does not exist \( x \in X \) such that \( f(x) \leq f(\hat{x}) \) (i.e., \( f_k(x) \leq f_k(\hat{x}), \) for \( k = 1,\ldots,p \)) and \( f(x) \neq f(\hat{x}) \). It is a weakly efficient (weakly Pareto optimal, weakly nondominated) solution for problem (1) if and only if there does not exists \( x \in X \) such that \( f(x) < f(\hat{x}) \). Let us denote \( X_E \) and \( X_{WE} \) the set of all efficient solutions and the set of all weakly efficient solutions for problem (1), respectively, so that \( X_E \subseteq X_{WE} \).

In many situations, the decision-making process does not require explicit enumeration of all efficient solutions, but only efficient solutions achieving the optimum of some scalar function expressing the decision maker’s preferences within the set of Pareto optimal solutions. This is the problem of optimizing over the efficient set, introduced by J. Phillip \cite{2}. This problem is given by

\[
\min_{x \in X_E} \phi(x)
\]

where \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is a real-valued function. The problem (2) is among the difficult problems in global optimization because the efficient set \( X_E \) is not convex in general, even in the linear case where the objective functions \( f_k \)'s are linear and the feasible set \( X \) is a polyhedron.

Most of the proposed methods to tackle this problem focus on the linear case. Several methods were developed following the work of J. Phillip \cite{2}; see for example the survey \cite{3} and the references therein. However, the literature dealing with this problem is relatively poor for nonlinear multi-objective problems due to its difficulty. Most of the algorithms proposed in the literature seek to globally solve this problem through global optimization techniques. It
is worth noting that such techniques are only able to solve moderate sized problems. In this paper, we propose a numerical method to tackle this problem when \( \phi(x) \) is a convex function, the objective functions \( f_k, k = 1, \ldots, p \), are quadratic convex and the feasible set \( X \) is nonempty, compact and convex. This algorithm is based on a penalty approach, using a sequence of convex nonlinear subproblems that can be solved efficiently. Our idea to tackle this problem is as follows. The problem (2) is reformulated as a nonlinear bilevel programming problem (NBLP) based on the characterization of the efficient solutions by means of scalar optimization problem derived from the multi-objective problem, and then to construct a penalty function that is a combination of the objective functions of the upper and the lower levels of the equivalent bilevel reformulation (NBLP). The resulting penalty problem is then solved by optimizing alternatively over the lower level variables and the upper level variables.

The proposed algorithm is shown to perform well on a set of standard problems from the literature, as it allows to obtain optimal solutions in all cases. In addition, it can be adapted easily to obtain a weakly efficient solution.

Références

